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Zeta-function Regularization, the Multiplicative Anomaly and the Wodzicki Residue

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Abstract: *The multiplicative anomaly associated with the zeta-function regularized determinant is computed for the Laplace-type operators $L_1 = -\Delta + V_1$ and $L_2 = -\Delta + V_2$, with V_1, V_2 constant, in a D-dimensional compact smooth manifold M_D, making use of several results due to Wodzicki and by direct calculations in some explicit examples. It is found that the multiplicative anomaly is vanishing for D odd and for $D = 2$. An application to the one-loop effective potential of the $O(2)$ self-interacting scalar model is outlined.*

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1 Introduction

Within the one-loop or external field approximation, the importance of zeta-function regularization for functional determinants, as introduced in [1], is well known, as a powerful tool to deal with the ambiguities (ultraviolet divergences) present in relativistic quantum field theory (see for example [2]-[4]). It permits to give a meaning, in the sense of analytic continuation, to the determinant of a differential operator which, as the product of its eigenvalues, is formally divergent. For the sake of simplicity we shall here restrict ourselves to scalar fields. The one-loop Euclidean partition function, regularised by zeta-function techniques, reads [5]

$$\ln Z = -\frac{1}{2}\ln\det\frac{L_D}{\mu^2} = \frac{1}{2}\zeta'(0|L_D) + \frac{1}{2}\zeta(0|L_D)\ln\mu^2\ ,$$

where $\zeta(s|L_D)$ is the zeta function related to L_D —typically an elliptic differential operator of second order— $\zeta'(0|L_D)$ its derivative with respect to s, and μ^2 a renormalization scale. The

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fact is used that the analytically continued zeta-function is generally regular at $s = 0$, and thus its derivative is well defined.

When the manifold is smooth and compact, the spectrum is discrete and one has

$$\zeta(s|L_D) = \sum_i \lambda_i^{-2s} ,$$

λ_i^2 being the eigenvalues of L_D. As a result, one can make use of the relationship between the zeta-function and the heat-kernel trace via the the Mellin transform and its inverse. For $\mathrm{Re}\, s > D/2$, one can write

$$\zeta(s|L_D) = \mathrm{Tr}\, L_D^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, K(t|L_D)\, dt\, , \tag{1.1}$$

$$K(t|L_D) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s > D/2} t^{-s}\, \Gamma(s) \zeta(s|L_D)\, ds\, , \tag{1.2}$$

where $K(t|L_D) = \mathrm{Tr} \exp(-tL_D)$ is the heat operator. The previous relations are valid also in the presence of zero modes, with the replacement $K(t|L_D) \longrightarrow K(t|L_D) - P_0$, P_0 being the projector onto the zero modes.

A heat-kernel expansion argument leads to the meromorphic structure of $\zeta(s|L_D)$ and, as we have anticipated, it is found that the analytically continued zeta-function is regular at $s = 0$ and thus its derivative is well defined. Furthermore, in practice all the operators may be considered to be trace-class. In fact, if the manifold is compact this is true and, if the manifold is not compact, the volume divergences can be easily factorized. Thus

$$K_t(L_D) = \int dV_D K_t(L_D)(x) \tag{1.3}$$

and

$$\zeta(L_D, z) = \int dV_D \zeta(L_D|z)(x), \tag{1.4}$$

where $K_t(L_D)(x)$ and $\zeta(L_D|z)(x)$ are the heat-kernel and the local zeta-function, respectively.

However, if an internal symmetry is present, the scalar field is vector valued, i.e. ϕ_i and the simplest model is the $O(2)$ symmetry associated with self-interacting charged fields in R^4. The Euclidean action is

$$S = \int dx^4 \phi_i \left[\left(-\Delta + m^2 \right) \phi_i + \frac{\lambda}{4!} (\phi^2)^2 \right] , \tag{1.5}$$

where $\phi^2 = \phi_k \phi_k$ is the $O(2)$ invariant. The Euclidean small disturbances operator reads

$$A_{ij} = L_{ij} + \frac{\lambda}{6} \Phi^2 \delta_{ik} + \frac{\lambda}{3} \Phi_i \Phi_k, \qquad L_{ij} = \left(-\Delta + m^2 \right) \delta_{ik}\, , \tag{1.6}$$

in which Δ is the Laplace operator and Φ the background field, assumed to be constant. Thus, one is actually dealing with a matrix-valued elliptic differential operator. In this case, the partition function is [6]

$$\ln Z = -\ln \det \left\| \frac{A_{ik}}{\mu^2} \right\| = -\ln \det \left[\frac{(L + \frac{\lambda}{2} \Phi^2)}{\mu^2} \frac{(L + \frac{\lambda}{6} \Phi^2)}{\mu^2} \right] \tag{1.7}$$

As a consequence, one has to deal with the product of two elliptic differential operators. In the case of a two-matrix, one has

$$\ln \det(AB) = \ln \det A + \ln \det B\, . \tag{1.8}$$

Usually the way one proceeds is by formally assuming the validity of the above relation for differential operators. This may be quite ambiguous, since one has to deal necessarily with a regularization procedure. In fact, it turns out that the zeta-function regularized determinants do *not* satisfy the above relation and, in general, there appears the so-called multiplicativity (or just multiplicative) anomaly [7, 8]. In terms of $F(A,B) \equiv \det(AB)/(\det A \det B)$ [8], it is defined as:

$$a_D(A,B) = \ln F(A,B) = \ln \det(AB) - \ln \det(A) - \ln \det(B) \,, \tag{1.9}$$

in which the determinants of the two elliptic operators, A and B, are assumed to be defined (e.g., regularized) by means of the zeta-function [1]. It should be noted that the non vanishing of the multiplicative anomaly implies that the relation

$$\ln \det A = \mathrm{Tr} \ln A \tag{1.10}$$

does not hold, in general, for elliptic operators like $A = BC$.

It turns out that this multiplicative anomaly can be expressed by means of the non-commutative residue associated with a classical pseudo-differential operator, known as the Wodzicki residue [9]. Its important role in physics has been recognized only recently. In fact, within the non-commutative geometrical approach to the standard model of the electroweak interactions [10, 11], the Wodzicki residue is the *unique* extension of the Dixmier trace (necessary to write down the Yang-Mills action functional) to the larger class of pseudo-differential operators (ΨDO) [12]. Other recent contributions along these lines are [13]-[15]. Furthermore, a proposal to make use of the Wodzicki formulae as a practical tool in order to determine the singularity structure of zeta-functions has appeared in [16] and the connection with the commutators anomalies of current algebras and the Wodzicki residue has been found in [17]

The purpose of the present paper is to obtain explicitly the multiplicative anomaly for the product of two Laplace-like operators —by direct computations and by making use of several results due to Wodzicki— and to investigate the relevance of these concepts in physical situations. As a result, the multiplicative anomaly will be found to be vanishing for D odd and also for $D = 2$, being actually present for $D > 2$, with D even.

The contents of the paper are the following. In Sect. 2 we present some elementary computations in order to show the highly non-trivial character of a brute force approach to the evaluation of the multiplicative anomaly associated with two differential operators (even with very simple ones). In Sect. 3 we briefly recall several results due to Wodzicki, concerning the noncommutative residue and a fundamental formula expressing the multiplicative anomaly in terms of the corresponding residue of a suitable pseudo-differential operator. In Sect. 4, the Wodzicki formula is used in the computation of the multiplicative anomaly in R^D and, as an example, the $O(2)$ model in R^4 is investigated. In Sect. 5, a standard diagrammatic analysis of the $O(2)$ model is discussed and evidence for the presence of the multiplicative anomaly at this diagrammatic level is given. In Sect. 6 we treat the case of an arbitrary compact smooth manifold without boundary. Some final remarks are presented in the Conclusions. In the Appendix a proof of the multiplicative anomaly formula is outlined.

2 Direct calculations

Motivated by the example discussed in the introduction, one might try to perform a direct computation of the multiplicative anomaly in the case of the two self-adjoint elliptic commuting operators $L_p = -\Delta + V_p$, $p = 1, 2$, in M_D, with V_p constant. Actually, we could deal with the shifts of two elliptic ΨODs. For the sake of simplicity, we may put $\mu^2 = 1$ and consider all

the quantities to be dimensionless. At the end, one can easily restore μ^2 by simple dimensional considerations.

In order to compute the multiplicative anomaly, one needs to obtain the zeta-functions of the operators. Let us begin with M_D smooth and compact without boundary (the boundary case can be treated along the same lines) and let us try to express $\zeta(s|L_1L_2)$ as a function of $\zeta(s|L_p)$. If we denote $L_0 = -\Delta$ and by λ_i its non-negative, discrete eigenvalues, the spectral theorem yields

$$\zeta(s|L_1L_2) = \sum_i [(\lambda_i + V_1)(\lambda_i + V_2)]^{-s} \ . \tag{2.1}$$

Making use of the identity

$$(\lambda_i + V_1)(\lambda_i + V_2) = (\lambda_i + V_+)^2 - V_-^2 \ , \tag{2.2}$$

with $V_+ = (V_1 + V_2)/2$ and $V_- = (V_1 - V_2)/2$, and noting that

$$\frac{V_-^2}{(\lambda_i + V_+)^2} < 1 \ , \tag{2.3}$$

for every individual λ_i, the binomial theorem gives

$$[(\lambda_i + V_1)(\lambda_i + V_2)]^{-s} = \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{k!\,\Gamma(s)} V_-^{2k} (\lambda_i + V_+)^{-2s-2k} \ , \tag{2.4}$$

an absolutely convergent series expansion, valid without further restriction. Let us assume that $\mathrm{Re}\, s$ is large enough in order to safely commute the sum over i with the sum over k. From the equations above, we get

$$\zeta(s|L_1L_2) = \zeta(2s|L_0 + V_+) + \sum_{k=1}^{\infty} \frac{\Gamma(s+k)}{k!\,\Gamma(s)} V_-^{2k} \zeta(2s + 2k|L_0 + V_+) \ . \tag{2.5}$$

This series is convergent for large $\mathrm{Re}\, s$ and provides the sought for analytical continuation to the whole complex plane.

To go further, we note that, when $|c| < \lambda_1$ (smallest non-vanishing eigenvalue of L), one has

$$\zeta(s|L + c) = \zeta(s|L) + \sum_{k=1}^{\infty} \frac{\Gamma(s+k)}{k!\,\Gamma(s)} (-c)^k \zeta(s + k|L) \ , \tag{2.6}$$

Let us use this expression for L_1 and L_2. Since

$$V_1 = V_+ + V_- \ , V_2 = V_+ - V_- \ , \tag{2.7}$$

one has

$$\zeta(s|L_1) = \zeta(s|L_0 + V_+ + V_-) = \zeta(s|L_0 + V_+) + \sum_{k=1}^{\infty} \frac{\Gamma(s+k)}{k!\,\Gamma(s)} (-V_-)^k \zeta(s + k|L_0 + V_+) \ , \tag{2.8}$$

and

$$\zeta(s|L_2) = \zeta(s|L_0 + V_+ - V_-) = \zeta(s|L_0 + V_+) + \sum_{k=1}^{\infty} \frac{\Gamma(s+k)}{k!\,\Gamma(s)} (V_-)^k \zeta(s + k|L_0 + V_+) \ . \tag{2.9}$$

For $s = 0$, there are poles, but adding the two zeta-functions for suitable $\mathrm{Re}\, s$ and making the separation between k odd and k even, all the terms associated with k odd cancel. As a result

$$\zeta(s|L_1) + \zeta(s|L_2) = 2\zeta(s|L_0 + V_+) + \sum_{m=1}^{\infty} \frac{\Gamma(s+2m)}{(2m)!\,\Gamma(s)} (V_-)^{2m} \zeta(s+2m|L_0+V_+)\,. \qquad (2.10)$$

For suitable $\mathrm{Re}\, s$, from Eqs. (2.5) and Eq. (2.10) we may write

$$\begin{aligned} \zeta(s|L_1L_2) - \zeta(s|L_1) - \zeta(s|L_2) &= \zeta(2s|L_0+V_+) - 2\zeta(s|L_0+V_+) \\ &+ \sum_{m=1}^{\infty} \frac{(V_-)^{2m}}{\Gamma(s)} \left[\frac{\Gamma(s+m)}{m!} \zeta(2s+2m|L_0+V_+) \right. \\ &- \left. 2\frac{\Gamma(s+2m)}{(2m)!} \zeta(s+2m|L_0+V_+) \right] , \end{aligned} \qquad (2.11)$$

The multiplicative anomaly is minus the derivative with respect to s in the limit $s \to 0$. Thus, it is present only when there are poles of the zeta functions evaluated at positive integer numbers bigger than 2. From the Seeley theorem, the meromorphic structure of the zeta function related to an elliptic operator is known, also in manifolds with boundary, the residues at the poles being simply related to the Seeley-De Witt heat-kernel coefficients A_r. For example, For a D-dimensional manifold without boundary one has [18]

$$\zeta(z|L) = \frac{1}{\Gamma(z)} \sum_{r=0}^{\infty} \frac{A_r}{z + r - \frac{D}{2}} + \frac{J(z)}{\Gamma(z)}\,, \qquad (2.12)$$

$J(z)$ being the analytical part. Since there are no poles at $s = 0$ for D odd and for $D = 2$ in the zeta functions appearing on the r.h.s. of Eq. (2.11), we can take the derivative at $s = 0$, i.e.

$$a_D(L_1, L_2) = \sum_{m=1}^{\infty} (V_-)^{2m} \zeta(2m|L_0+V_+) \left[\frac{\Gamma(m)}{\Gamma(m+1)} - 2\frac{\Gamma(2m)}{\Gamma(2m+1)} \right] . \qquad (2.13)$$

As a consequence, for D odd and for $D = 2$ the multiplicative anomaly is vanishing.

For $D > 2$ and even, there are a finite number of simple poles other than at $s = 0$ in Eq. (2.11). As an example, in the important case $D = 4$, in a compact manifold without boundary, the zeta function has simple poles at $s = 2, s = 1, s = 0$, etc. Only the first one is relevant, the other being harmless. Separating the term corresponding to $l = 1$, only this gives a non vanishing contribution when one takes the derivatives with respect to s at zero. Thus, a direct computation yields

$$a_4(L_1, L_2) = \frac{A_0 V_-^2}{2} = \frac{\mathcal{V}_D}{4(4\pi)^2} (V_1 - V_2)^2\ . \qquad (2.14)$$

It follows that it exists potentially, an alternative direct method for computing the multiplicative anomaly for the shifts of two elliptic ΨDOs and its structure will be a function of V_-^2 and of the heat-kernel coefficients A_r, which, in principle, are computable (the first ones are known). We will come back on this point in Sect. 6, using the Wodzicki formula.

However, we observe that, here, the multiplicative anomaly is a function of the series of zeta-functions related to operators of Laplace type. One soon becomes convinced that it is not easy to go further along this way for an arbitrary D-dimensional manifold.

We conclude this section with explicit examples.

Example 1: $M_D = R^D$.

Let us start with a particularly simple example, i.e. $M_D = R^D$. The two zeta-functions $\zeta(s|L_i)$ are easy to evaluate and read

$$\zeta(s|L_i) = \frac{\mathcal{V}_D}{(4\pi)^{\frac{D}{2}}} V_i^{\frac{D}{2}-s} \frac{\Gamma(s-\frac{D}{2})}{\Gamma(s)}, \qquad i = 1, 2\,, \tag{2.15}$$

where $\mathcal{V}_D$ is the (infinite) volume of R^D. We need to compute $\zeta(s|L_1L_2)$. For $\mathrm{Re}\, s > D/2$, starting from the spectral definition, one gets

$$\zeta(s|L_1L_2) = \frac{2\mathcal{V}_D}{4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2})} \int_0^\infty dk k^{D-1} \left[k^4 + (V_1+V_2)k^2 + V_1V_2\right]^{-s} . \tag{2.16}$$

For $\mathrm{Re}\, s > (D-1)/4$, the the above integral can be evaluated [19], to yield

$$\zeta(s|L_1L_2) = \frac{\sqrt{2}\pi\mathcal{V}_D\Gamma(2s-\frac{D}{2})}{2^s(4\pi)^{\frac{D}{2}}\Gamma(s)} \left(\alpha^2-1\right)^{\frac{1-2s}{4}} (V_1V_2)^{\frac{D}{4}-s} P^{\frac{1}{2}-s}_{s-\frac{D+1}{2}}(\alpha)\,, \tag{2.17}$$

$P^\mu_\nu(z)$ being the associate Legendre function of the first kind (see for example [19]), and

$$\alpha = \frac{V_1+V_2}{2\sqrt{V_1V_2}}\,. \tag{2.18}$$

This provides the analytical continuation to the whole complex plane. For $D = 2Q+1$, one easily gets

$$\begin{aligned}
\zeta(0|L_1L_2) &= 0, \\
\zeta'(0|L_1L_2) &= \frac{\sqrt{2}\pi\mathcal{V}_D\Gamma(-Q-\frac{1}{2})}{(4\pi)^{\frac{D}{2}}} \left(\alpha^2-1\right)^{\frac{1}{4}} (V_1V_2)^{\frac{D}{4}} P^{\frac{1}{2}}_{-\frac{D+1}{2}}(\alpha) \\
&= \frac{\mathcal{V}_D\Gamma(-Q-\frac{1}{2})}{(4\pi)^{\frac{D}{2}}} \left[2(V_1V_2)^{\frac{D}{2}}(1+\cosh(D\gamma))\right]^{1/2}\,,
\end{aligned} \tag{2.19}$$

in which $\cosh\gamma = \alpha$. The first equation says that the conformal anomaly vanishes. On the other hand, one has for D odd

$$\zeta'(0|L_1) + \zeta'(0|L_2) = \frac{\mathcal{V}_D\Gamma(-Q-\frac{1}{2})}{(4\pi)^{\frac{D}{2}}} \left(V_1^{\frac{D}{2}} + V_2^{\frac{D}{2}}\right)\,, \tag{2.20}$$

As a consequence, making use of elementary properties of the hyperbolic cosine, one gets $a(L_1, L_2) = 0$. Namely, for D odd the multiplicative anomaly is vanishing (see [8]).

For $D = 2Q$, the situation is much more complex. First the conformal anomaly is non-zero, i.e.

$$\zeta(0|L_1L_2) = \frac{\mathcal{V}_D}{(4\pi)^Q} \frac{(-1)^Q}{Q!} \left[(V_1V_2)^{Q/2}\cosh(Q\gamma)\right]\,, \tag{2.21}$$

and, in general, the multiplicative anomaly is present. As a check, for $D = 2$, we get

$$\zeta(0|L_1L_2) = -\frac{\mathcal{V}_2}{4\pi}\left[(V_1V_2)^{1/2}\cosh\gamma\right] = -\frac{\mathcal{V}_2}{4\pi}(V_1+V_2) = \frac{1}{4\pi}a_1(A) = \zeta(0|A)\,, \tag{2.22}$$

where $A = -\Delta I + V$ is a 2×2 matrix-valued differential operator, I the identity matrix, $V = \mathrm{diag}\,(V_1, V_2)$, and $a_1(A)$ is the first related Seeley-De Witt coefficient, given by the well known expression $\int dx^2(-\mathrm{tr}\, V)$.

Unfortunately, it is not simple to write down —within this naive approach— a reasonably simple expression for it, because the associate Legendre function depends on s through the two indices μ and ν. However, it is easy to show that the anomaly is absent when $V_1 = V_2$, therefore it will depend only on the difference $V_1 - V_2$. Thus, one may consider the case $V_2 = 0$. As a result, Eq. (2.16) yields the simpler expression

$$\zeta(s|L_1 L_2) \quad = \quad \frac{\sqrt{2}\pi \mathcal{V}_D}{(4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2})} \frac{\Gamma(\frac{D}{2}-s)\Gamma(2s-\frac{D}{2})}{\Gamma(s)} V_1^{\frac{D}{2}-2s} \,. \tag{2.23}$$

In this case the multiplicative anomaly is given by

$$a(L_1, L_2) = \ln\det(L_1 L_2) - \ln\det(L_1)\,, \tag{2.24}$$

since the regularized quantity $\ln\det(L_2) = 0$. It is easy to show that, when D is odd, again $a_D(L_1, L_2) = 0$. When $D = 2Q$, one obtains

$$a_{2Q}(L_1, L_2) = \frac{\mathcal{V}_D}{(4\pi)^Q} \frac{(-1)^Q}{2Q!} V_1^Q \left[\Psi(1) - \Psi(Q)\right] \,. \tag{2.25}$$

We conclude this first example by observing that the multiplicative anomaly is absent when $Q = 1$, $D = 2$, and that it is present for $Q > 1$, $D > 2$ even. The result obtained is partial and more powerful techniques are necessary in order to deal with the general case. Such techniques will be introduced in the next section.

Example 2: $M_D = S^1 \times R^{D-1}$, $D = 1, 2, 3, \dots$

In this case the zeta functions corresponding to L_i, $i = 1, 2$, are given by

$$\zeta(s|L_i) = \frac{\pi^{(D-1)/2-2s}\Gamma(s+(1-D)/2)}{2^{2s+1}L^{D-2s}\Gamma(s)} \sum_{n=-\infty}^{\infty} \left[n^2 + \left(\frac{L}{2\pi}\right)^2 V_i\right]^{(D-1)/2-s} \tag{2.26}$$

($i = 1, 2$, here L is the length of S^1). In terms of the basic zeta function (see [20]):

$$\begin{aligned} \zeta(s;q) \quad &\equiv \quad \sum_{n=-\infty}^{\infty} (n^2+q)^{-s} \qquad\qquad (2.27) \\ &= \sqrt{\pi}\, \frac{\Gamma(s-1/2)}{\Gamma(s)} q^{1/2-s} + \frac{4\pi^s}{\Gamma(s)} q^{1/4-s/2} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n\sqrt{q}), \end{aligned}$$

where K_ν is the modified Bessel function of the second kind, we obtain

$$\begin{aligned} \zeta(s|L_i) \quad &= \quad \frac{\pi^{-D/2}}{\Gamma(s)} \Big[2^{-D} L \Gamma(s-D/2) V_i^{D/2-s} \\ &\quad + 2^{2-s-D/2} L^{s+1-D/2} V_i^{D/4-s/2} \sum_{n=1}^{\infty} n^{s-D/2} K_{s-D/2}(nL\sqrt{V_1}) \Big] \qquad (2.28) \\ &\equiv \zeta^{(1)}(s|L_i) + \zeta^{(2)}(s|L_i). \end{aligned}$$

For the determinant we get, for D odd,

$$\begin{aligned} \det L_i \quad &= \quad \exp\Big\{ -\pi^{-D/2} \Big[2^{-D} L \Gamma(-D/2) V_i^{D/2} \\ &\quad + (2L)^{1-D/2} V_i^{D/4} \sum_{n=1}^{\infty} n^{-D/2} K_{D/2}(nL\sqrt{V_i}) \Big] \Big\}, \qquad (2.29) \end{aligned}$$

for D even ($D=2Q$),

$$\det L_i = \exp\left[-\frac{L}{Q!}\left(-\frac{1}{4\pi}\right)^Q V_i^Q\left(\sum_{j=1}^Q\frac{1}{j}-\ln V_i\right)\right.$$
$$\left.+\,4L\left(\frac{\sqrt{V_i}}{2\pi L}\right)^Q\sum_{n=1}^\infty n^{-Q}K_Q(nL\sqrt{V_i})\right]. \qquad (2.30)$$

As for the product L_1L_2, using the same strategy as before, after some calculations we obtain (here we use the short-hand notation $L_\pm \equiv L_0+V_\pm$, cf. equations above):

$$\det(L_1L_2) = (\det L_+)^2\exp\left\{-\sum_{p=1}^{[Q/2]}2L\frac{V_-^{2p}(-V_+)^{Q-2p}}{(2p)!(Q-2p)!(4\pi)^Q}\right.$$
$$\times\left[1-C+\frac{1}{2\,(Q-2p)!}+\frac{1}{2}\sum_{j=1}^{p-1}\frac{1}{j}-\psi(2p)-\ln V_+\right]$$
$$\left.-\sum_{p=[Q/2]+1}^\infty\frac{V_-^{2p}}{p}\zeta^{(1)}(2p|L_+)-\sum_{p=1}^\infty\frac{V_-^{2p}}{p}\zeta^{(2)}(2p|L_+)\right\}, \qquad (2.31)$$

where $[x]$ means 'integer part of x' and C is the Euler-Mascheroni constant. We can check from these formulas that the anomaly (1.9) is zero in the case of odd dimension D. Actually, this is most easily seen, as before, by using the expression corresponding to (2.16) for the present case. It also vanishes for $D=2$. The formula above is useful in order to obtain numerical values for the case D even, corresponding to different values of D and L (the series converge very quickly). The results are given in Table 1. We have looked at the variation of the anomaly in terms of the different parameters: L, D, V_1 and V_2 while keeping the rest of them fixed. Within numerical errors, we have checked the complete coincidence with formula (4.5) in Sect. 4.

Example 3: $M_D = R^D$ with Dirichlet b.c. on p pairs of perpendicular hyperplanes.

The zeta function is, in this case,

$$\zeta(s|L_i) = \frac{\pi^{(D-p)/2-2s}\Gamma(s+(p-D)/2)}{2^{D-p+1}\prod_{j=1}^p a_j\,\Gamma(s)}\sum_{n_1,\ldots,n_p=1}^\infty\left[\sum_{j=1}^p\left(\frac{n_j}{a_j}\right)^2+V_i\right]^{(D-p)/2-s}, \qquad (2.32)$$

where the a_j, $j=1,2,\ldots,p$, are the pairwise separations between the perpendicular hyperplanes. For the determinant, we get, for $D-p=2h+1$ odd,

$$\det L_i = \exp\left\{-\frac{\pi^{h+1/2}}{2^{2h+2}\prod_{j=1}^p a_j}\Gamma(-h-1/2)\sum_{n_1,\ldots,n_p=1}^\infty\left[\sum_{j=1}^p\left(\frac{n_j}{a_j}\right)^2+V_i\right]^{h-1/2}\right\}, \qquad (2.33)$$

and, for $D-p=2h$ even,

$$\det L_i = \exp\left\{\frac{(-\pi)^h}{2^{2h+1}h!\prod_{j=1}^p a_j}\left[\left(2+h\sum_{j=1}^{h-1}\frac{1}{j}\right)\sum_{n_1,\ldots,n_p=1}^\infty\left[\sum_{j=1}^p\left(\frac{n_j}{a_j}\right)^2+V_i\right]^h\right.\right.$$
$$\left.\left.+\sum_{n_1,\ldots,n_p=1}^\infty\left[\sum_{j=1}^p\left(\frac{n_j}{a_j}\right)^2+V_i\right]^h\ln\left[\sum_{j=1}^p\left(\frac{n_j}{a_j}\right)^2+V_i\right]\right]\right\}. \qquad (2.34)$$

For the calculation of the anomaly one follows the same steps of the two preceding examples and we are not going to repeat this again. In order to obtain the final numbers one must make use of the inversion formula for the Epstein zeta functions of these expressions [20, 2].

L	D	V_1	V_2	$a(L_1, L_2)$
1	2	2	2	0.
0.1	2	8	3	-1.8686×10^{-14}
1	2	8	3	-2.0817×10^{-17}
5	2	8	3	-1.4572×10^{-16}
10	2	8	3	-1.4572×10^{-16}
1	2	10	1	2.87×10^{-12}
1	4	10	1	0.064117
1	6	10	1	–0.028063
1	8	10	1	0.0151245
1	10	10	1	–0.003636
1	12	10	1	0.0006124
1	14	10	1	–0.00008166
1	16	10	1	9.09×10^{-6}
1	4	2	1	0.0007916
1	4	5	2	0.007124
1	4	1	6	0.019789
1	6	2	1	–0.0000945
1	6	5	2	–0.001984
1	6	1	6	–0.005512
0.1	4	7	2	0.001979
0.5	4	7	2	0.009895
1	4	7	2	0.019789
2	4	7	2	0.0395786
5	4	7	2	0.098947
10	4	7	2	0.197893
20	4	7	2	0.395786
0.1	6	7	2	–0.00070865
0.5	6	7	2	–0.00354326
1	6	7	2	–0.0070865
2	6	7	2	–0.014173
5	6	7	2	–0.0354326
10	6	7	2	–0.07008652
20	6	7	2	–0.141730

Table 1: Values of the multiplicative anomaly $a(L_1, L_2)$ in terms of the parameters: L, D, V_1 and V_2. Observe its evolution when some of the parameters are kept fixed while the others are varied. In all cases, a perfect coincidence with Wodzicki's expression for the anomaly is obtained (within numerical errors).

3 The Wodzicki residue and the multiplicative anomaly

For reader's convenience, we will review in this section the necessary information concerning the Wodzicki residue [9] (see, also [7] and the references to Wodzicki quoted therein) that will be used in the rest of the paper. Let us consider a D-dimensional smooth compact manifold without boundary M_D and a (classical) ΨDO, A, of order m, acting on sections of vector bundles on M_D. To any ΨDO, A, it corresponds a complete symbol $a(x,k)$, such that, modulo infinitely smoothing operators, one has

$$(Af)(x) \sim \int_{R^D} \frac{dk}{(2\pi)^D} \int_{R^D} dy e^{i(x-y)k} a(x,k) f(y) \,. \tag{3.1}$$

The complete symbol admits an asymptotic expansion for $|k| \to \infty$, given by

$$a(x,k) \sim \sum_j a_{m-j}(x,k) \,, \tag{3.2}$$

and fulfills the homogeneity property $a_{m-j}(x,tk) = t^{m-j} a_{m-j}(x,k)$, for $t > 0$. The number m is called the order of A.

If P is an elliptic operator of order $p > m$, according to Wodzicki one has the following property of the non-commutative residue, which we may take as its characterization.

Proposition. The trace of the operator AP^{-s} exists and admits a meromorphic continuation to the whole complex plane, with a simple pole at $s = 0$. Its Cauchy residue at $s = 0$ is proportional to the so-called non-commutative (or Wodzicki) residue of A:

$$\text{res}(A) = p \, \text{Res}_{s=0} \, \text{Tr}(AP^{-s}) \,. \tag{3.3}$$

The r.h.s. of the above equation does not depend on P and is taken as the definition of the Wodzicki residue of the ΨDO, A.

Properties. (i) Strictly related to the latter result is the one which follows, involving the short-t asymptotic expansion

$$\text{Tr}(Ae^{-tP}) \simeq \sum_j \alpha_j t^{\frac{D-j}{p}-1} - \frac{\text{res}(A)}{p} \ln t + O(t \ln t) \,. \tag{3.4}$$

Thus, the Wodzicki residue of A, a ΨDO, can be read off from the above asymptotic expansion selecting the coefficient proportional to $\ln t$.

(ii) Furthermore, it is possible to show that res(A) is linear with respect to A and possesses the important property of being the unique trace on the algebra of the ΨDOs, namely, one has $\text{res}(AB) = \text{res}(BA)$. This last property has deep implications when including gravity within the non-commutative geometrical approach to the Connes-Lott model of the electro-weak interaction theory [12, 10, 11].

(iii) Wodzicki has also obtained a local form of the non-commutative residue, which has the fundamental consequence of characterizing it through a scalar density. This density can be integrated to yield the Wodzicki residue, namely

$$\text{res}(A) = \int_{M_D} \frac{dx}{(2\pi)^D} \int_{|k|=1} a_{-D}(x,k) dk \,. \tag{3.5}$$

Here the component of order $-D$ of the complete symbol appears. Form the above result it immediately follows that $\text{res}(A) = 0$ when A is an elliptic differential operator.

(iv) We conclude this summary with the multiplicative anomaly formula, again due to Wodzicki. A more general formula has been derived in [8]. Let us consider two invertible elliptic

self-adjont operators, A and B, on M_D. If we assume that they commute, then the following formula applies

$$a(A,B)=\frac{\text{res}\left[(\ln(A^bB^{-a}))^2\right]}{2ab(a+b)}=a(B,A)\,, \tag{3.6}$$

where $a>0$ and $b>0$ are the orders of A and B, respectively. A sketch of the proof is presented in the Appendix. It should be noted that $a(A,B)$ depends on a ΨDO of zero order. As a consequence, it is independent on the renormalization scale μ appearing in the path integral.

(v) Furthermore, it can be iterated consistently. For example

$$\begin{aligned}\zeta'(A,B) &= \zeta'(A)+\zeta'(B)+a(A,B), \qquad (3.7)\\ \zeta'(A,B,C) &= \zeta'(AB)+\zeta'(C)+a(AB,C)=\zeta'(A)+\zeta'(B)+\zeta'(C)+a(A,B)+a(AB,C)\,.\end{aligned}$$

As a consequence,

$$a(A,B,C)=a(AB,C)+a(A,B)\,. \tag{3.8}$$

Since $a(A,B,C)=a(C,B,A)$, we easily obtain the cocycle condition (see [8]):

$$a(AB,C)+a(A,B)=a(CB,A)+a(C,B)\,. \tag{3.9}$$

4 The $O(2)$ bosonic model

In this section we come back to the problem of the exact computation of the multiplicative anomaly in the model considered in Sect. 2. Strictly speaking, the result of the last section is valid for a compact manifold, but in the case of R^D the divergence is trivial, being contained in the volume factor. The Wodzicki formula gives

$$a(L_1,L_2)=\frac{1}{8}\,\text{res}\left[(\ln(L_1L_2^{-1}))^2\right]\,. \tag{4.1}$$

We have to construct the complete symbol of the ΨDO of zero order $[\ln(L_1L_2^{-1})]^2$. It is given by

$$a(x,k)=\left[\ln(k^2+V_1)-\ln(k^2+V_2)\right]^2\,. \tag{4.2}$$

For large k^2, we have the following expansion, from which one can easily read off the homogeneuos components:

$$a(x,k)=\sum_{j=2}^{\infty}c_jk^{-2j}=\sum_{j=2}^{\infty}a_{2j}(x,k)\,, \tag{4.3}$$

where

$$c_j=\sum_{n=1}^{j}\frac{(-1)^j}{n(j-n)}\left(V_1^n-V_2^n\right)\left(V_1^{j-n}-V_2^{j-n}\right)\,. \tag{4.4}$$

As a consequence, due to the local formula one immediately gets the following result: for D odd, the multiplicative anomaly vanishes, in perfect agreement with the direct calculation of Sect. 2. This result is consistent with a general theorem contained in [8].

For D even, if $D = 2$ one has no multiplicative anomaly, while for $D = 2Q$, $Q > 1$, one gets

$$a(L_1, L_2) = \frac{\mathcal{V}_D(-1)^Q}{4(4\pi)^Q\Gamma(Q)} \sum_{j=1}^{Q-1} \frac{1}{j(Q-j)} \left(V_1^j - V_2^j\right)\left(V_1^{Q-j} - V_2^{Q-j}\right) . \tag{4.5}$$

It is easy to show that for $V_2 = 0$ this expression reduces to the one obtained directly in Sect. 2.

In the $O(2)$ model, for $D = 4$, we have

$$a(L_1, L_2) = \frac{\mathcal{V}_4}{4(4\pi)^2}(V_1 - V_2)^2 = \frac{\mathcal{V}_4}{36(4\pi)^2}\lambda^2\Phi^4 , \tag{4.6}$$

which, for dimensional reasons, is independent of the renormalization parameter μ. Then, the one-loop effective potential reads

$$V_{eff} = -\frac{\ln Z}{\mathcal{V}_4} = \frac{M_1^4}{64\pi^2}\left(-\frac{3}{2} + \ln\frac{M_1^2}{\mu^2}\right) + \frac{M_2^4}{64\pi^2}\left(-\frac{3}{2} + \ln\frac{M_2^2}{\mu^2}\right) + \frac{1}{72(4\pi)^2}\lambda^2\Phi^4 , \tag{4.7}$$

with

$$M_1^2 = m^2 + \frac{\lambda}{2}\Phi^2, \qquad M_2^2 = m^2 + \frac{\lambda}{6}\Phi^2 . \tag{4.8}$$

Thus, the additional multiplicative anomaly contribution seems to modify the usual Coleman-Weinberg potential. A more careful analysis is required in order to investigate the consequences of this remarkable fact.

5 Feynmann diagrams

The necessity of the presence of the multiplicative anomaly in quantum field theory can also be understood perturbatively, using the background field method. The effective action of the $O(2)$ model in a background field Φ will be denoted by $\Gamma(\Phi, \phi)$, where ϕ is the mean field. Then, if $\Gamma_0(\phi)$ denotes the effective action with vanishing Φ, it turns out that

$$\Gamma(\Phi, \phi) = \Gamma_0(\Phi + \phi). \tag{5.1}$$

Therefore, the n-th order derivatives of Γ with respect to ϕ at $\phi = 0$ determine the vertex functions of the $O(2)$ model in the background external field. The one-loop approximation to Γ is again given by $\log\det(L_1L_2)$, and the determinant of either of the operators, L_1 and L_2, corresponds to the sum of all vacuum-vacuum 1PI diagrams where only particles of masses squared $M_1^2 = m^2 + \lambda\Phi^2/2$ or $M_2^2 = m^2 + \lambda\Phi^2/6$ flow along the internal lines. In Fig. 1 we have depicted this, by using a solid line for type-1 particles and a dashed line for type-2 particles.

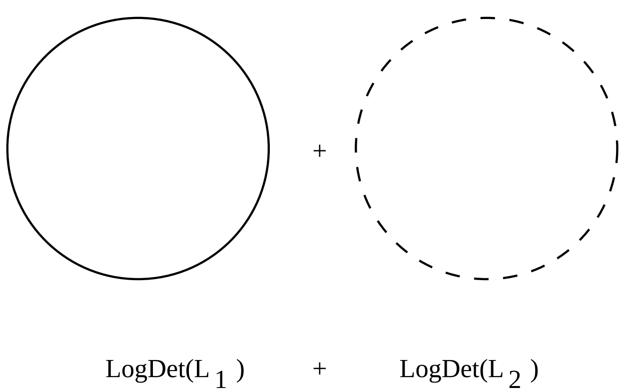

Figure 1. The Feynmann graph giving the one-loop effective potential without taking into account the anomaly.

Thus, for example, the inverse propagator at zero momentum for type-1 particle, as computed from the above effective potential, is obtained from the second derivative with respect to ϕ_1. The only 1PI graphs which contribute are shown in Fig. 2.

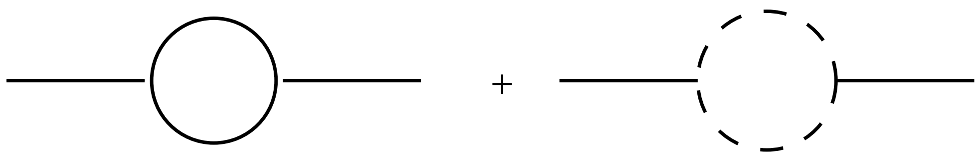

Figure 2. Contributions coming from 1PI graphs.

This is clearly not the case, as the full theory exhibits a trilinear coupling $\phi_2(\phi_1)^2$ which gives the additional Feynmann graph depicted in Fig. 3.

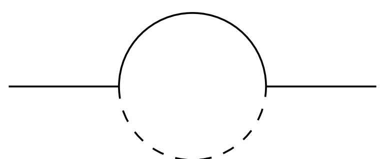

Figure 3. Additional Feynman graph of the full theory.

Without investigating this question any further, we can safely affirm already that a perturbative formula for the Wodzicki anomaly given in terms of Feynmann diagrams should exist. It surely owes its simple form to very subtle cancellations among an infinite class of Feynmann diagrams.

We conclude this section with some remarks. In the present model, the existence of a multiplicative anomaly of the type considered could be a trivial problem, in fact it has the same form as the classical potential energy. This suggests that it can be absorbed in a finite renormalization of the coupling constant of the theory. Secondly, this anomaly gives no contribution to the one-loop beta function of the model, since it is independent of the arbitrary renormalization scale, but it certainly contributes to the two-loop beta function. And, finally, we have seen that the anomaly can be interpreted as an external field effect which, in the present model, could be relevant only when the theory is coupled to an external source. Therefore, it should be very interesting to study its relevance in at least two other situations, namely the cases of a spontaneously broken symmetry and of QED in external background fields.

6 The case of a general, smooth and compact manifold M_D without boundary

Since the multiplicative anomaly is a local functional, it is possible to express it in terms of the Seeley-De Witt spectral coefficients. Let us consider again the operator $L_p = L_0 + V_p$, with $L_0 = -\Delta$ acting on scalars, in a smooth and compact manifold M_D without boundary. We have to compute the Wodzicki residue of the ΨDO

$$\left[\ln(L_1 L_2^{-1})\right]^2 . \tag{6.1}$$

With this aim, if $V_1 < V_2$, we can consider the ΨDO

$$\left[\ln(L_1 L_2^{-1})\right]^2 e^{-tL_1} \,, \tag{6.2}$$

and compute the $\ln t$ term in the short-t asymptotic expansion of its trace. We are dealing here with self-adjoint operators and thus, by using the spectral theorem, we get

$$\mathrm{Tr}\left[\left(\ln(L_1 L_2^{-1})\right)^2 e^{-tL_1}\right] = \int_{V_1}^{\infty} d\lambda \rho(\lambda|L_1) \left[\ln\lambda - \ln(\lambda + V_2 - V_1)\right]^2 e^{-t\lambda} \,, \tag{6.3}$$

where $\rho(\lambda|L_1)$ is the spectral density of the self-adjoint operator L_1.

Now, it is well known that the short-t expansion of the above trace receives contributions from the asymptotics, for large λ, of the integrand in the spectral integral. The asymptotics of the spectral function associated with L_1 are known to be given by (see, for example [21, 22], and the references therein)

$$\rho(\lambda|L_1) \simeq \sum_{r=0}^{r<D/2} \frac{A_r(L_1)}{\Gamma(\frac{D}{2} - r)} \lambda^{\frac{D}{2}-r-1} \,, \tag{6.4}$$

here the quantities $A_r(L_1)$ are the Seeley-De Witt heat-kernel coefficients while, for large λ, we have in addition

$$\left[\ln\lambda - \ln(\lambda + V_2 - V_1)\right]^2 \simeq \sum_{j=2}^{\infty} b_j \lambda^{-j} \,, \tag{6.5}$$

being the b_j computable, for instance $b_2 = (V_2 - V_1)^2$, $b_3 = -2(V_2 - V_1)^3$, etc. As a result, we get the short-t asymptotics in the form

$$\mathrm{Tr}\left[\left(\ln L_1 L_2^{-1}\right)^2 e^{-tL_1}\right] \simeq \sum_{r=0}^{r<D/2} \frac{A_r(L_1)}{\Gamma(\frac{D}{2} - r)} \sum_{j=2}^{\infty} b_j t^{r+j-\frac{D}{2}} \Gamma(\tfrac{D}{2} - r - j, tV_1) \,, \tag{6.6}$$

where $\Gamma(z, x)$ the incomplete gamma function. From this expression one obtains the following results:

(i) If D is odd, say $D = 2Q + 1$, the first argument of the incomplete gamma function is never zero or a negative integer. Thus, the $\ln t$ is absent and, from the Wodzicki theorem, the multiplicative anomaly is absent too, again in agreement with the Kontsevich-Vishik theorem [8] and the explicit calculations in the previous sections.

(ii) If D is even, we have to search for the log terms only, that is $-Q + r + j = 0$, for $r \geq 0$ and $j \geq 2$. As a result, for $D = 2$ the log term is absent once more, again in agreement with the explicit calculations of the previous sections. The multiplicative anomaly is present starting from $D \geq 4$. In the important case when $D = 4$, it turns out that the multiplicative anomaly is identical to the one, related with R^4, that has been evaluated previosly. Terms depending on the curvature become operative only for $D \geq 6$.

7 Conclusions

In this paper, the multiplicative anomaly associated with the zeta-function regularised determinant of two ΨDOs of Laplace type on a D-dimensional smooth manifold without boundary has been studied. From a physical point of view, this condition does not seem to be too restrictive, because the one-loop effective potential may be expressed as a logarithm of the determinant of such kind of elliptic differential operators.

We have shown how a direct calculation leads to analytical difficulties, even in the most simple examples. Fortunately, a very elegant formula for the multiplicative anomaly has been found by Wodzicki and we have used it here in order to compute the anomaly explicitly. It is worth mentioning that, from a computational point of view, this constitutes a big improvement, since one can make use of the results concerning the computation of one-loop effective potential, related to second order elliptic differential operators of Laplace type. Furthermore, within the background field method, we have identified the presence of the multiplicative anomaly in the diagrammatic perturbative approach too.

With regard to our example, namely the product L_1L_2, we have shown that the multiplicative anomaly is vanishing for D odd and also for $D = 2$. This seems to be related with the fact that we have only considered differential operators of second order (Laplace type). For first-order differential operators (Dirac like), things could be quite different, in principle, and we will consider this important case elsewhere.

Another interesting issue is the generalization of all these procedures to smooth manifolds with a boundary. Again one should expect to obtain different results in those situations.

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A Appendix: The Wodzicki formula for the multiplicative anomaly

In this Appendix, for the reader's convenience we present a proof of the multiplicative anomaly formula along the lines of Ref. [8].

Recall that if P is an elliptic operator of order $p > a$, according to Wodzicki, one has the following property of the non-commutative residue related to the ΨDO A: in a neighborhood of $z = 0$, it holds

$$z\,\mathrm{Tr}(AP^{-z}) = \frac{1}{\Gamma(1+z)}\frac{\mathrm{res}(A)}{p} + O(z^2)\,. \tag{A.1}$$

Now we resort to the following

Lemma. If η is a ΨDO of zero order, a, and B a ΨDO of positive order, b, and γ and x positive real numbers then, in a neighborhood of $s = 0$, one has

$$s\,\mathrm{Tr}(\ln\eta\,\eta^{-xs}B^{-\gamma s}) = \frac{\mathrm{res}(\ln\eta)}{\Gamma(1+\gamma s)\gamma b} - sx\frac{\mathrm{res}((\ln\eta)^2)}{\Gamma(1+\gamma s)\gamma b} + O(s^2)\,. \tag{A.2}$$

The Lemma is a direct consequence of the formal expansion

$$\eta^{-xs} = e^{-xs\ln\eta} = I - xs\ln\eta + O(s^2) \tag{A.3}$$

and of Eq. (A.1). From the above Lemma, it follows that

$$\lim_{s\to 0}\partial_s\left[s\,\mathrm{Tr}(\ln\eta\,\eta^{-xs}B^{-\gamma s})\right] = C\frac{\mathrm{res}(\ln\eta)}{b} - x\frac{\mathrm{res}[(\ln\eta)^2]}{\gamma b}\,, \tag{A.4}$$

in which C is the Euler-Mascheroni constant.

Now consider two invertible, commuting, elliptic, self-adjont operators A and B on M_D, with a and b being the orders of A and B, respectively. Within the zeta-function definition of the determinants, consider the quantity

$$F(A,B) = \frac{\det(AB)}{(\det A)(\det B)} = e^{a(A,B)} \,. \tag{A.5}$$

Introduce then the family of ΨDOs

$$A(x) = \eta^x B^{\frac{a}{b}} \,, \qquad \eta = A^b B^{-a} \,, \tag{A.6}$$

and define the function

$$F(A(x),B) = \frac{\det(A(x)B)}{(\det A(x))(\det B)} \,. \tag{A.7}$$

One gets

$$F(A(0),B) = \frac{\det B^{\frac{a+b}{b}}}{(\det B^{\frac{a}{b}})(\det B)} = 1 \,, \qquad F(A(\tfrac{1}{b}),B) = \frac{\det(AB)}{(\det A)(\det B)} = F(A,B) \,. \tag{A.8}$$

As a consequence, one is led to deal with the following expression for the anomaly

$$a(A(x),B) = \ln F(A(x),B) = -\lim_{s\to 0} \partial_s \left[\mathrm{Tr}(A(x)B)^{-s} - \mathrm{Tr}\, A(x)^{-s} - \mathrm{Tr}\, B^{-s}\right] \,. \tag{A.9}$$

This quantity has the properties: $a(A(0),B) = 0$ and $a(A(\frac{1}{b}),B) = a(A,B)$.

The next step is to compute the first derivative of $a(A(x),B)$ with respect to x, the result being

$$\partial_x a(A(x),B) = \lim_{s\to 0} \partial_s \left[\mathrm{Tr}\left(\ln\eta\,\eta^{-xs} B^{-s\frac{a+b}{b}}\right) - \mathrm{Tr}\left(\ln\eta\,\eta^{-xs} B^{-s\frac{a}{b}}\right)\right] \,. \tag{A.10}$$

Making now use of Eq. (A.4), one obtains

$$\begin{aligned} \partial_x a(A(x),B) &= C\frac{\mathrm{res}(\ln\eta)}{b} - x\frac{\mathrm{res}[(\ln\eta)^2]}{a+b} - C\frac{\mathrm{res}(\ln\eta)}{b} + x\frac{\mathrm{res}[(\ln\eta)^2]}{a} \\ &= x\frac{b}{a(a+b)}\mathrm{res}[(\ln\eta)^2] \,. \end{aligned} \tag{A.11}$$

And, finally, performing the integration with respect to x, from 0 to $1/b$, one gets Wodzicki's formula for the multiplicative anomaly, used in Sect. 3, namely

$$a(A,B) = a(B,A) = \frac{\mathrm{res}\left[(\ln(A^b B^{-a}))^2\right]}{2ab(a+b)} \,. \tag{A.12}$$

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